Towards a mathematical definition of Coulomb branches of 3-dimensional N=4 gauge theories

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Algebraic Lie Theory and Representation Theory 2015

20150606/07

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§0. motivation

G_c: compact Lie group (G: its complexification)
M: a quaternionic representation (symplectic representation of G)

$\underset{\text{Physics}}{\rightsquigarrow}$ 3d $N=4$ SUSY gauge theory associated with (G_c, M)

$\underset{\text{Physics}}{\rightsquigarrow}$ Moduli Space of vacua

It has two distinguished branches

- $\mathcal{M}_H$: Higgs branch
- $\mathcal{M}_C$: Coulomb branch

hyperKähler manifolds with $SU(2)$-action
(rotating cpx structures)

$\mathcal{M}_H$: mathematically rigorous defined:

$$\mathcal{M}_H = M /\!/\!/ G_c \quad : \text{hyperKähler quotient}$$

$$= \vec{\mu}^{-1}(0)/G_c \qquad \underset{\text{h.K. moment map}}{\vec{\mu} : M \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*} \qquad = \mu_{\mathbb{C}}^{-1}(0)/\!/G \quad \text{symplectic reduction}$$

$\vec{\mu} = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})$

But there is no mathematically rigorous definition of $\mathcal{M}_C$.

Physicists have many examples of $\mathcal{M}_C$
(or many recipe to determine $\mathcal{M}_C$)

e.g.
- moduli space of magnetic monopoles on $\mathbb{R}^3$
- " instantons on $\mathbb{R}^4$ etc
- nilpotent orbits of type A (and conjecturally for classical groups) at least
  $\cap$ Slodowy slice

<u>Today</u> Assume $M = N \oplus N^*$ (as $G_{\mathbb{C}}$-module)
(also G : connected)

Give a definition of $\mathcal{M}_C$ as an affine variety (scheme)

$$\mathcal{M}_C = \operatorname{Spec} \mathcal{A}$$

with many interesting properties
structures

e.g. quantization, integrable systems etc

§1. Examples

∘ $N=0$ $\qquad \mathcal{M}_H = \{0\}$, but $\mathcal{M}_C$ nontrivial (discussed later)

∘ toric hyperKähler

$$\begin{tikzcd} 1 \arrow[r] & T^\ell \arrow[r] \arrow[d, equal] & \tilde{T}=(\mathbb{C}^\times)^n \arrow[r] \arrow[d, phantom, "\circlearrowright"] & T_F=(\mathbb{C}^\times)^{n-\ell} \arrow[r] & 1 \\ & G & \mathbb{C}^n & & \end{tikzcd}$$

natural action

$$\rightsquigarrow \quad \mathcal{M}_H = \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!/\!/ T_{\mathbb{C}}^\ell$$

$$\mathcal{M}_C = \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!/\!/ T^\vee_{F,\mathbb{C}}$$

$T_F^\vee$: dual torus $\subset \tilde{T}^\vee \cong \tilde{T} \curvearrowright \mathbb{C}^n$

∘ $N=\mathfrak{g}$: adjoint representation

$$\rightsquigarrow \quad \mathcal{M}_H = \mathfrak{g}\oplus\mathfrak{g}^* /\!/\!/ G_{\mathbb{C}} = \{(\alpha,\beta)\in \mathfrak{g}\times \overset{\overset{\mathfrak{g}}{\wr\!\wr}}{\mathfrak{g}^*} \mid [\alpha,\beta]=0\} /\!/ G = \mathfrak{t}\times\mathfrak{t}^*/W$$

$$\mathcal{M}_C \overset{?}{=} T^*T^\vee/W = \mathfrak{t}\times T^\vee/W$$

$T^\vee$ = dual torus of $T\subset G$

cf. Vasserot's construction of DAHA
on equivariant K-theory of the affine Steinberg variety

spherical part of degenerate DAHA, as we use
equivariant homology of the affine Grassmannian Steinberg variety

- quiver gauge theory

$Q = (Q_0, Q_1)$: quiver of type ADE $\qquad$ G_Q = corresponding simple group (adjoint type)

V : Q_0-graded vector space

$$G = \prod_i GL(V_i) \curvearrowright N = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{o(h)}, V_{i(h)})$$

$$\rightsquigarrow \mathcal{M}_H = N \oplus N^* /\!/\!/ G_c = \{0\} \quad \text{(Lusztig)}$$

$\mathcal{M}_C$ = moduli space of G_Q-monopoles on $\mathbb{R}^3$ with charge $= \overrightarrow{\dim} V$

= moduli space of based maps $\mathbb{P}^1 \longrightarrow$ flag of degree $= \overrightarrow{\dim} V$

V, W : Q_0-graded

G : same $\qquad N = \bigoplus_h \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_i \mathrm{Hom}(V_i, W_i)$

$\rightsquigarrow$ $\mathcal{M}_H = N \oplus N^* /\!/\!/ G_c$: quiver variety (of type ADE)

$\mathcal{M}_C \overset{?}{=}$ moduli space of G_Q-monopoles on $\mathbb{R}^3$ with charge $= \overrightarrow{\dim} V$

singularity at 0 with type $= \overrightarrow{\dim} W$

= slices in affine Grassmannian

(If $\mu = \sum \dim W_i \cdot \Lambda_i - \sum \dim V_i \, \alpha_i$: dominant)

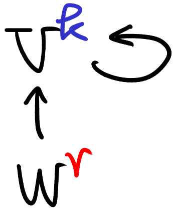
- subexample: Q: type A $\Longrightarrow$ $\mathcal{M}_H \cong \mathcal{S}_\lambda \cap \overline{\mathcal{N}_\mu}$

$\mathcal{M}_C \overset{?}{\cong} \mathcal{S}_{\mu^t} \cap \overline{\mathcal{N}_{\lambda^t}}$

$\overline{\mathcal{N}_\mu}$: nilpotent orbit closure

$\mathcal{S}_\lambda$: Slodowy slice

○ Jordan quiver



⇝ $\mathcal{M}_H$ = Uhlenbeck space for $U(r)$-instantons on $\mathbb{R}^4$ with change k

$\mathcal{M}_C = S^k(\mathbb{R}^4/\mathbb{Z}_r)$: Uhlenbeck for "$U(1)$-instantons" on $\mathbb{R}^4/\mathbb{Z}_r$ with change k

(NB quantization of $\mathcal{M}_C$: spherical part of cyclotomic rational Cherednick algebra)

More generally

$\mathcal{M}_H$ = quiver variety of affine ADE type level $= \langle \overrightarrow{\dim} W, \delta \rangle = r$

⇝ $\mathcal{M}_C$ = Uhlenbeck space for G_Q-instantons on $\mathbb{R}^4/\mathbb{Z}_r$ with charge & repr. at 0 given by $\overrightarrow{\dim} V$

§2. Definition of $\mathcal{M}_C$

- Reminder of affine Grassmannian and [BFM]
- Step 1°. An infinite dimensional variety $\mathcal{R} \equiv \mathcal{R}_{G,N}$
- 2° Convolution product on $H_*^{G_\mathcal{O}}(\mathcal{R}_{G,N})$ $G_\mathcal{O} = G[[z]]$

$G_\mathcal{K} = G((z))$, $G_\mathcal{O} = G[[z]]$ $D = \operatorname{Spec} \mathbb{C}[[z]] \supset D^\times = \operatorname{Spec} \mathbb{C}((z))$

$\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$: affine Grassmann (ind-scheme)

$\cong \{ (\mathcal{P}, \varphi) \mid \mathcal{P} : G\text{-bdle over } D,\ \varphi : \mathcal{P}|_{D^\times} \xrightarrow{\cong} G \times D^\times \text{ trivialization over } D^\times \}$

convolution diagram for Gr_G :

$$\begin{tikzcd}
\mathrm{Gr}_G \times \mathrm{Gr}_G & G_\mathcal{K} \times \mathrm{Gr}_G \arrow[l, "p"'] \arrow[r, "q"] & G_\mathcal{K} \times_{G_\mathcal{O}} \mathrm{Gr}_G \arrow[r, "m"] & \mathrm{Gr}_G \\
([g_1], [g_2]) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & (g_1, [g_2]) \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & {[g_1, [g_2]]} \arrow[r, mapsto] \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & {[g_1 g_2]} \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

☆ geometric Satake : $A_1, A_2 \in \mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$

$$A_1 * A_2 := m_*(q^*)^{-1} p^*(A_1 \boxtimes A_2)$$

$(\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G), *)$: tensor category $\cong (\mathrm{Rep}\, G^\vee, \otimes)$

☆ [Bezrukavnikov-Finkelberg-Mirković]

$H_*^{G_{\mathcal{O}}}(\mathrm{Gr}_G) \ni c_1, c_2$ $\qquad c_1 * c_2 := m_*(q^*)^{-1} p^*(c_1 \boxtimes c_2)$

$H_*^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ is a graded commutative algebra with $1 = [e]$

— noncommutative deformation $H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ ← loop rotation

— integrable system $H^*_{G_{\mathcal{O}}}(pt) = H^*_G(pt) \longrightarrow H_*^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$

$\wr$ polynomial ring (if G: connected)

Th. [BFM]

$\mathrm{Spec}\, H_*^{G_\mathcal{O}}(\mathrm{Gr}_G) \longrightarrow \mathrm{Spec}\, H_G^*(pt) = \mathbb{C}^\ell$ is the Kostant-Toda system

for $G^\vee$ = Langlands dual group

$T^*G^\vee \curvearrowleft G^\vee \times G^\vee$ left-right multiplication

$\langle \underset{\mathfrak{n}_+^\vee}{\underset{\uparrow}{e}}, \underset{\mathfrak{n}_-^\vee}{\underset{\uparrow}{f}}, h \rangle$: $\mathfrak{sl}_2$-triple for regular nilpotent element

$N_\pm^\vee$: unipotent group

$\mu_{N_-}^\vee : T^*G^\vee \longrightarrow (\mathfrak{n}_-^\vee \oplus \mathfrak{n}_-^\vee)^*$: moment map for $N_-^\vee \times N_-^\vee$ -action

Kostant-Toda lattice $= \mu_{N_-}^{\vee\,-1}(e,e)/_{N_-^\vee \times N_-^\vee} \longrightarrow \nu^{-1}(e)/_{N_-^\vee} \cong e + \mathfrak{z}(f) \cong \mathfrak{t}/_W$

Kostant slice $\nu : \mathfrak{g}^{\vee *} \to \mathfrak{n}_-^{\vee *}$

NB. $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$: quantum Hamiltonian reduction of Diff $G^\vee$

— This is the special case $N=0$. $(\Rightarrow \mathcal{M}_H = \{0\})$

2nd day

G : connected reductive group

$G_K = G((z)) \supset G_\mathcal{O} = G[[z]]$

$\mathrm{Gr}_G = G_K / G_\mathcal{O}$: affine Grassmannian

$$\begin{tikzcd} \mathrm{Gr}_G \times \mathrm{Gr}_G & G_K \times \mathrm{Gr}_G \arrow[l, "p"'] \arrow[r, "q"] & G_K \times^{G_\mathcal{O}} \mathrm{Gr}_G \arrow[r, "m"] & \mathrm{Gr}_G \end{tikzcd}$$

$H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ is an associative algebra by the convolution product

$$c_1 * c_2 := m_*(q^*)^{-1} p^*(c_1 \boxtimes c_2)$$

- $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ is graded, commutative

← loop rotation

- $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$: noncommutative deformation
- $$\begin{tikzcd} H_G^*(pt) \arrow[r] \arrow[d, equal] & H_*^{G_\mathcal{O}}(\mathrm{Gr}_G) \\ \mathbb{C}[\mathfrak{g}/\!/\mathrm{Ad}G] = \mathbb{C}[\mathfrak{t}]^W & \end{tikzcd}$$

Th [BFM] $\quad \operatorname{Spec} H_*^{G_O}(Gr_G) \longrightarrow \mathfrak{t}/W \cong \mathbb{C}^\ell$

is the Kostant-Toda integrable system for $G^\vee$: Langlands dual group

Hamiltonian reduction of $T^*G^\vee$ by $N_-^\vee \times N_-^\vee$ $\qquad N_-^\vee \subset G^\vee$

unipotent

NB. $H_*^{G_O \rtimes \mathbb{C}^\times}(Gr_G)$ is the quantized Hamiltonian reduction of $\mathrm{Diff}(G^\vee)$

◎ general N

$Gr_G = G_K / G_O$ as before

$$\begin{tikzcd}
\mathcal{T}\equiv\mathcal{T}_{G,N} := \arrow[d, "\substack{\infty\text{-rank}\\ \text{vector}\\ \text{bundle}}"] & G_K \times_{G_O} N_O \arrow[r, "f"] & N_K \\
Gr_G & \rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\
 & {[g,s]} \arrow[r, mapsto] & gs
\end{tikzcd}$$

$$\boxed{\mathcal{R}\equiv\mathcal{R}_{G,N} := f^{-1}(N_O)}$$

Rem. $St = T^*\mathcal{F} \times_{(\text{nilp. var})} T^*\mathcal{F} \curvearrowleft G$

$= \{(\mathcal{B}_1, x, \mathcal{B}_2) \in \mathcal{F}\times\mathfrak{g}\times\mathcal{F} \mid x \in \text{nilrad}(\text{Lie}\,\mathcal{B}_1) \cap \text{nilrad}(\text{Lie}\,\mathcal{B}_2)\}$

Fix $\mathcal{B}_1 = \mathcal{B}$ $\quad \supset \{(x, \mathcal{B}_2) \in \mathfrak{g}\times\mathcal{F} \mid x \in \text{nilrad}(\text{Lie}\,\underset{\substack{\uparrow\\ \text{fixed}}}{\mathcal{B}}) \cap \text{nilrad}(\text{Lie}\,\mathcal{B}_2)\} = \overline{St}$

$$G\backslash St = B\backslash\overline{St}$$

Our $\mathcal{R}$ is an analog of $\overline{St}$

St version : $\mathcal{T} \times_{N_K} \mathcal{T}$

(too ∞-dimensional to work)

$\mathrm{Gr}_G = \bigsqcup_{\lambda} \mathrm{Gr}_{G,\lambda}$ (λ: dominant coweight)

↖ $G_{\mathcal{O}}$-orbit (finite dimensional, smooth)

$$\begin{tikzcd}
\mathcal{R} \arrow[r] & \mathrm{Gr}_G \\
\mathcal{T} \arrow[ru] &
\end{tikzcd}$$

$\mathcal{R}_\lambda$, $\mathcal{T}_\lambda$ = inverse image of $\mathrm{Gr}_{G,\lambda}$

☆ $\mathcal{R}_\lambda \longrightarrow \mathrm{Gr}_{G,\lambda}$ is a vector bundle (of ∞-rank)
subbundle of $\mathcal{T}_\lambda$ s.t. $\mathcal{T}_\lambda / \mathcal{R}_\lambda$: finite rank

Consider equivariant Borel-Moore homology group $H_*^{G_{\mathcal{O}}}(\mathcal{R})$.

cycles s.t — finite dimensional in base-direction
— finite codimensional in fiber-direction (relative to $\mathcal{T}$)

The grading is $\mathbb{Z}$-valued (not $\mathbb{Z}_{\geq 0}$)

◎ convolution product $*$ is defined by a similar diagram as in Gr_G.

<u>NB</u> $\quad \mathrm{St} = T^*F \times_{\mathcal{N}} T^*F \Lleftarrow T^*F \times_{\mathcal{N}} T^*F \times_{\mathcal{N}} T^*F$

§3. Properties of $\mathcal{A}$ and $\mathcal{M}_C$

$\mathcal{A} := H_*^{G_\mathcal{O}}(\mathcal{R})$ + convolution product $\qquad \mathcal{M}_C := \operatorname{Spec} \mathcal{A}$

1) $\mathcal{A}$ is a $\mathbb{Z}$-graded algebra (finitely generated)
(So $\mathcal{M}_C$ has a $\mathbb{C}^\times$-action.)

unit 1 = fundamental class of fiber over $[e] \in Gr_G$

2) $\mathcal{A}$ has a "natural" noncommutative deformation $\mathcal{A}_\hbar$ over $\mathbb{C}[\hbar]$
by $\mathcal{A}_\hbar = H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R})$

Then $\{\ ,\ \} = \frac{1}{\hbar}[\ ,\ ]\big|_{\hbar=0}$: Poisson bracket on $\mathcal{A}$ $(\deg = -1$
or $-2)$

3) filtration

$$Gr_G = \bigsqcup_{\substack{\lambda:\text{ dominant}\\ \text{coweight}}} Gr_{G,\lambda} = \bigcup \underset{\substack{\uparrow\\ \text{closure}}}{Gr_{G,\le\lambda}} \rightsquigarrow \mathcal{R} = \bigcup \mathcal{R}_{\le\lambda}$$

<u>Claim.</u> Mayer-Vietories splits $\rightsquigarrow \mathcal{A} = H_*^{G_\mathcal{O}}(\mathcal{R}) = \bigcup H_*^{G_\mathcal{O}}(\mathcal{R}_{\le\lambda})$
associated graded $\operatorname{gr}\mathcal{A} = \bigoplus_\lambda H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda)$

⊙ $\mathrm{gr}\mathcal{A}$ has an explicit presentation. $\mathcal{M}_C \underset{\text{degenerate}}{\rightsquigarrow}$ something combinatorial

$\uparrow$

Q. What is this?

$$\begin{tikzcd}
\mathcal{R}_\lambda \arrow[r] \arrow[d, phantom, sloped, "\subset"] & Gr_{G,\lambda} \arrow[r] & G/P_\lambda \\
\mathcal{T}_\lambda & &
\end{tikzcd}$$

both vector b'dles

$$\rightsquigarrow \quad H^{G_\mathcal{O}}_*(\mathcal{R}_\lambda) \cong H^{G_\mathcal{O}}_{*-2\mathrm{rank}(\mathcal{T}_\lambda/\mathcal{R}_\lambda)}(Gr_{G,\lambda})$$

$$\cong H^{*+\text{explicit}}_{\mathrm{Stab}(\lambda)}(pt)$$

$$\mathbb{C}[\mathfrak{t}]^{W_\lambda}$$
$$\cong$$

$\Longrightarrow$ $\bigoplus H^{G_\mathcal{O}}_*(\mathcal{R}_\lambda)$ has a base $\{f_p[\mathcal{R}_\lambda]\}$ f_p : base of $H^*_{\mathrm{Stab}(\lambda)}(pt)$

moreover multiplication is

$$f_p[\mathcal{R}_\lambda] \times g_f[\mathcal{R}_\mu] = f_p g_f a_{\lambda,\mu}[\mathcal{R}_{\lambda+\mu}]$$

where

$$a_{\lambda,\mu} = \begin{cases} \prod_{\substack{\nu:\text{ wt of } N \\ \neq 0}} \nu^{\mathrm{mult}(\nu)\cdot\min(|\langle\lambda,\nu\rangle|, |\langle\mu,\nu\rangle|)} & \text{if } \langle\lambda,\nu\rangle\langle\mu,\nu\rangle < 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\begin{tikzcd} 4) & & H^*_G(pt) \arrow[r] \arrow[dl, phantom, "\cong" sloped] & H^{G_O}_*(\mathcal{R}) \\ & \mathbb{C}[\mathfrak{g}]^{AdG} & c \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & c\cap 1 \arrow[u, phantom, "\in" sloped] \\ & \mathbb{C}[\mathfrak{t}]^W \arrow[u, phantom, "\cong" sloped] & & \end{tikzcd}$$

gives $\mathcal{M}_C \xrightarrow[\text{flat}]{\Phi} \mathfrak{t}/W \cong \mathbb{C}^\ell$

$\mathfrak{t} = \mathrm{Lie}\, T$

$T \subset G$: max. torus

W : Weyl group

<u>Claim</u>. Over generic point in $\mathfrak{t}/W$ (Spec (Frac $H^*_G(pt)$))
(more precisely over the complement of finite union of hyperplanes in $\mathfrak{t}/W$)

$$\begin{tikzcd} \mathcal{M}_C|_{\mathring{\mathfrak{t}}/W} \arrow[r, phantom, "\cong"] \arrow[dr, "\Phi"'] & T^*T^\vee/W\,\Big|_{\mathring{\mathfrak{t}}/W} = \mathfrak{t}\times T^\vee/W\,\Big|_{\mathring{\mathfrak{t}}/W} \arrow[d, "p_1"] \\ & \mathring{\mathfrak{t}}/W \subset \mathfrak{t}/W \end{tikzcd}$$

(Therefore the integrable system is solved already)

proof) $H^{G_O}_*(\mathcal{R}) \otimes_{H^*_G(pt)} \mathrm{Frac}\, H^*_T(pt) \underset{\text{localization}}{\cong} H_*(\mathcal{R}^T) \otimes_{\mathbb{C}} \mathrm{Frac}\, H^*_T(pt)$

Then $\mathcal{R}^T = Gr_T \times N^T_{\mathcal{O}}$ $\quad$ N^T = T-fixed part of N

$Gr_T \cong$ coweight lattice of T

$\therefore$ Spec $H_*(\mathcal{R}^T) \cong T^\vee$: dual torus //

quantization : $\mathbb{C}[\mathfrak{t},\hbar]^W \hookrightarrow \mathcal{A}_\hbar$: quantized Coulomb branch

Claim. Commutative subalgebra !!

(called ~~Gelfand-Tsetlin~~ Cartan subalgebra)

$\therefore$ Φ : Poisson commute. Hence Φ : integrable system

4) $\mathcal{M}_C$ has an action of $\pi_1(G)^\wedge$: Pontryagin dual of $\pi_1(G)$

$\Leftrightarrow$ $\mathbb{C}[\mathcal{M}_C]$ has a $\pi_1(G)$-grading

In fact, $\pi_0(\mathcal{R}) = \pi_0(\mathrm{Gr}_G) = \pi_1(G)$ $\therefore$ $\mathcal{R} = \bigsqcup_{\gamma \in \pi_1(G)} \mathcal{R}_\gamma$

$$H_*^{G_\mathcal{O}}(\mathcal{R}_\gamma) * H_*^{G_\mathcal{O}}(\mathcal{R}_{\gamma'}) \longrightarrow H_*^{G_\mathcal{O}}(\mathcal{R}_{\gamma+\gamma'})$$

NB. G : semisimple $\Longrightarrow$ $\pi_1(G)$: finite abelian group

$\pi_1(G)^\vee$ is also

$G = T^\ell \Rightarrow \pi_1(G) \cong \mathbb{Z}^\ell$ $\therefore$ $\pi_1(G)^\vee = T^\ell$

$G = GL \Rightarrow \pi_1(G) \cong \mathbb{Z}$ $\qquad \pi_1(G)^\vee = \mathbb{C}^\times$

5) flavor symmetry

Suppose $\exists$

$1 \longrightarrow G \longrightarrow \tilde{G} \longrightarrow G_F \longrightarrow 1$ s.t. $M = N \oplus N^*$ is a $\tilde{G}$-module

(e.g. $G_F = \prod GL(W_i) \times H_1(\text{graph})$ for quiver gauge theory)

$\Rightarrow$ $\mathcal{M}_C$ has a commutative deformation over $\mathfrak{g}_F /\!/ \mathrm{Ad}\, G_F$

In fact, $\mathcal{R}$ has a $\tilde{G}_{\mathcal{O}}$-action

$\rightsquigarrow$ $H_*^{\tilde{G}_{\mathcal{O}}}(\mathcal{R}) \longleftarrow H^*_{\tilde{G}}(pt) \longleftarrow H^*_{G_F}(pt)$ $\downarrow$ Spec

$\tilde{\mathcal{M}}_C \longrightarrow \tilde{\mathfrak{g}} /\!/ \mathrm{Ad}\, \tilde{G} \longrightarrow \mathfrak{g}_F /\!/ \mathrm{Ad}\, G_F$ fiber over 0 = original $\mathcal{M}_C$

One can also construct a (partial) resolution of singularities for each dominant coweight $\lambda_F : \mathbb{C}^* \longrightarrow G_F$

In fact, consider $\tilde{\mathcal{R}} = \mathcal{R}_{\tilde{G}, N} \longrightarrow \mathrm{Gr}_{\tilde{G}} \longrightarrow \mathrm{Gr}_{G_F}$ fiber over $[e] \in \mathrm{Gr}_{G_F}$ = original $\mathcal{R}$

Use the stratification on Gr_{G_F}, to introduce a filtration on $H_*^{\tilde{G}_{\mathcal{O}}}(\tilde{\mathcal{R}})$. Then take the associated graded.

Braden-Licata-Proudfoot-Webster : Symplectic duality

$$\text{quantization of } \mathcal{M}_C \xleftrightarrow[\text{Koszul dual}]{} \text{quantization of } \mathcal{M}_H$$

under some conditions ($\mathcal{M}_C$ was not defined in [BLPW])

§4 (Conjectural) "duality" between $\mathcal{M}_H$ and $\mathcal{M}_C$. (More elementary than [BLPW])

1) stratum

Fact. $\mathcal{M}_H$ has a stratification (symplectic leaves)

$$\mathcal{M}_H = \coprod_{\alpha \in A} \mathcal{M}_H^\alpha$$

$A = \{$conjugacy classes of stabilizers$\}$

Conjecture $\mathcal{M}_C$ has a stratification parametrized by the same set A : $\mathcal{M}_C = \coprod_{\alpha \in A} \mathcal{M}_C^\alpha$

with the opposite closure relation

(e.g., $\mathcal{M}_H = \{0\} \Rightarrow \mathcal{M}_C$: smooth symplectic manifold)

moduli space of vacua $= \coprod_{\alpha \in A} \mathcal{M}_C^\alpha \times \mathcal{M}_H^\alpha$

2) $\mathbb{C}^\times$-actions

$\mathbb{C}^\times \curvearrowright \underset{t}{M}$ or $\underset{t}{N} \oplus \underset{1}{N^*}$ $\rightsquigarrow$ $\mathbb{C}^\times \curvearrowright \mathcal{M}_H$

Then $\mathbb{C}[\mathcal{M}_H] = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}[\mathcal{M}_H]_n$ & grade $0 = \mathbb{C}$ (often) (i.e. $\mathcal{M}_H$ is cone)

On the other hand, $\mathcal{M}_C$ is not cone in general

Conjecture $\mathcal{M}_C$ is cone $\underset{?}{\Longleftrightarrow}$ $\mu_{\mathbb{C}}^{-1}(0) \subset M$ is complete intersection

3) Group action and deformation/resolution
(mass parameter) (Kähler parameter)

- flavor symmetry $1 \to G \to \tilde{G} \to G_F \to 1$

$\mathrm{Hom}(\mathbb{C}^*, G_F) \ni \lambda_F \rightsquigarrow \mathbb{C}^\times \xrightarrow{\lambda_F} G_F \curvearrowright \mathcal{M}_H = M /\!/\!/ G$

$\rightsquigarrow$ deformation / resolution of $\mathcal{M}_C$

- $\mathrm{Hom}_{grp}(G, \mathbb{C}^*) \ni \chi \rightsquigarrow \mu_{\mathbb{C}}^{-1}(0) /\!/_{\chi} G = \mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^{G, \chi^n}\right) \longrightarrow \mathcal{M}_H$

<u>Note</u> $\cong$ often (partial) resolution

$\mathrm{Hom}_{grp}(\mathbb{C}^*, \pi_1(G)^\wedge) \ni \chi \rightsquigarrow \mathbb{C}^\times \xrightarrow{\chi} \pi_1(G)^\wedge \curvearrowright \mathcal{M}_C$ group action

Thus mass / Kähler parameter are exchanged between $\mathcal{M}_C$ and $\mathcal{M}_H$.

<u>Conjecture</u> $\underset{\chi}{\lambda_F}$ has fixed points only $\{0\}$ on $\underset{\mathcal{M}_C}{\mathcal{M}_H}$

$\Longleftrightarrow$ $\underset{\chi}{\lambda_F}$ gives a resolution (orbifold in general) of $\underset{\mathcal{M}_H}{\mathcal{M}_C}$